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ON THE PLASTIC LOADING PROCESS BEHIND AN UNLOADING SHOCK FRONT*

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The problem of elastic wave refraction in an elastic-plastic half-space (EPH) in the active loading domain has been investigated /1-4/ for different models of an elastic-plastic body. The problem has been solved /5/ for refraction of a pure shear elastic wave that has a profile of steps of finite length in an EPH in both the active plastic loading domain and in the unloading zone under the assumption that the material behind the unloading shock (US) is in the elastic state. It is shown below for this problem that a plastic loading process can be realized behind the US front and a solution is constructed in the secondary plastic flow zone.

1. A medium is under antiplane deformation conditions when a pure shear wave propagates. The displacement velocity vector w is directed along the x_3 axis and depends on the variables x_1, x_2 and the time t, and the stresses $\tau_1 = \sigma_{13}(x_1, x_2, t)$, $\tau_2 = \sigma_{23}(x_1, x_2, t)$ differ from zero. Henceforth we will confine ourselves to investigating selfsimilar solutions of the equations of the dynamics of an ideal elastic-plastic body that depend on two variables $x = x_1 - ct$ and $y = x_2$. In this case the equations of the characteristics and the relationships along the characteristics of the system of motion equations have the following form /3/:

in the elastic domain and unloading zone

$$x + \varkappa y = \text{const}, \ \varkappa w - \tau_2 = \text{const} \tag{1.1}$$

$$x - \varkappa y = \text{const}, \ \varkappa w + \tau_2 = \text{const} \tag{1.2}$$

$$y = \text{const}, \ \tau_1 + w = f(y), \ \varkappa = \sqrt{M^2 - 1}, \ M = c/a, \ a = \sqrt{\mu/\rho} \tag{1.3}$$

in the active plastic loading domain

$$dy\,(M + \cos\theta) = -\sin\theta\, dx, \ \theta + Mw = \text{const} \tag{1.4}$$

$$dy\,(M - \cos\theta) = \sin\theta\, dx, \ \theta - Mw = \text{const}, \ \tau_1 = \sin\theta, \quad \tau_2 = \cos\theta \tag{1.5}$$

Here ρ is the density and μ is the shear modulus.

Equations (1.1)-(1.5) are written in dimensionless variables that will be used later (to simplify the writing the bars above the dimensionless variables are omitted, and k is the yield point)

$$\bar{x} = \frac{x}{l}, \quad \bar{y} = \frac{y}{l}, \quad \bar{\tau}_1 = \frac{\tau_1}{k}, \quad \bar{\tau}_2 = \frac{\tau_2}{k}, \quad \bar{w} = \frac{w}{w^*}, \quad w^* = \frac{ck}{\mu}$$

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2. Let a pure shear plane wave OA with a profile of steps of finite length (Fig.1) be incident from an elastic half-space $y<0$, characterized by the parameters $\mu_1, \rho_1, a_1 = \sqrt{\mu_1/\rho_1}$, on the interfacial boundary $y=0$ with the EPH $y>0$, whose mechanical properties are defined by the parameters $\mu_2, \rho_2, a_2 = \sqrt{\mu_2/\rho_2}, k$. Because of the interaction of the incident wave OA with the interfacial boundaries a refracted wave OC and a reflected wave OB are formed. The EPH material ahead of the wave front OC is at rest and there are no initial stresses therein. It is assume that the stress τ_2 on the interfacial boundary and the rate of displacement w are continuous, whence we have /5/

$$2w_1(-x\sin\varphi_1) = w(x) - \mu \operatorname{tg}\varphi_1\tau_2(x), \quad \mu = \mu_2/\mu_1 \tag{2.1}$$

Here $w(x)$ and $\tau_2(x)$ are the displacement velocity and the stress on the interfacial boundary in the EPH, $w_1(-x\sin\varphi_1)$ is a function yielding the incident wave profile and intensity ($w_1(-x\sin\varphi_1) = W_0 = \text{const}$ for $x_N \leqslant x\sin\varphi_1 \leqslant 0$ and $w_1(-x\sin\varphi_1) = 0$ for $x\sin\varphi_1 < x_N$), and φ_1 is the angle of incidence.

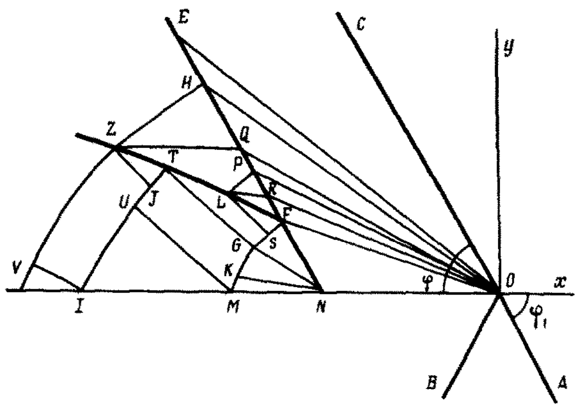

Fig.1

The wave pattern in the EPH is shown in Fig.1. The material is in the elastic state in the domain EOC while plastic deformation of the EPH material occurs in the domain EON and NE is the US. It has been shown /5/ that the plasticity condition behind the US front can be satisfied on the interfacial boundary for the problem under consideration. Consequently, we assume that the EPH material in a certain neighbourhood of the point N behind the US front is in the plastic state. The following relationships hold on the line of strong discontinuity NE /5/.

$$\tau_1^- = \tau_1^+ + [w], \quad \tau_2^- = \tau_2^+ + \varkappa[w], \quad \varkappa = \sqrt{M^2-1}, \quad M = c/a_2 \tag{2.2}$$

Here $[\tau] = \tau^+ - \tau^-$, τ^+, τ^- are the limiting values of τ on the US front in the plastic loading and unloading domains respectively.

The stresses τ_1^- and τ_2^- satisfy the flow condition

$$\tau_1^{-2} + \tau_2^{-2} = 1 \tag{2.3}$$

We determine the jump in w on the US from (2.2) and (2.3)

$$[w] = -2(\tau_1^+ + \varkappa\tau_2^+)M^{-2} \tag{2.4}$$

We obtain for the stresses from (1.5), (2.2) and (2.4)

$$\tau_1^- = \sin(\theta^+ - 2\varphi), \quad \tau_2^- = \cos(\theta^+ - 2\varphi) \tag{2.5}$$

The plasticity condition (2.3) will be satisfied if we set

$$\tau_1^- = \sin\theta^-, \quad \tau_2 = \cos\theta^- \tag{2.6}$$

We determine the jump in θ on the US from (2.5) and (2.6)

$$[\theta] = \theta^+ - \theta^- = 2\theta^+ - \pi - 2\varphi \tag{2.7}$$

Relationships (1.4) and (1.5) hold in the plastic loading domain. But $\theta^+ \in [\pi, \pi+\varphi]$ /5/ (φ is the angle of refraction), and hence we obtain from (2.7) that $\theta^- \in [\varphi, 2\varphi]$. Therefore, the characteristics (1.5) intersect the line NF and the relationships

$$dy(M - \cos\theta) = \sin\theta\, dx, \quad \theta - Mw = \theta^- - Mw^- \tag{2.8}$$

are satisfied thereon.

A quantity that is constant for all the characteristics of this family that intersect the segment NF is on the right side of the second equality in (2.8) (since the quantities θ^+, w^+ are constant on the NF /5/). Consequently, we obtain from (1.4) and (2.8) that w and θ do not vary along the characteristics of the other family, from which it follows that the characteristics (1.4) are rectilinear

$$y(M+\cos\theta)+\sin\theta\,(x-x_N)=0,\ \theta+Mw=\mathrm{const} \tag{2.9}$$

The characteristics (2.9) are inclined to the x axis at an angle ψ_0 which determines the position of the characteristic NG (Fig.1) ($\operatorname{tg}\psi_0=-\sin\theta_M^-/(M+\cos\theta_N^-)<\operatorname{tg}\varphi$). The stresses and displacement velocity are constant in the domain GNF

$$\tau_1=\sin\theta_N^-,\ \tau_2=\cos\theta_N^-,\ w=M^{-1}(1+\pi+\varphi-\theta_N^+ + 2\cos(\theta^+-\varphi),\ \theta_N^-=2\varphi+\pi-\theta_N^+ \tag{2.10}$$

and relationships (2.8) take the form

$$dy\,(M-\cos\theta)=\sin\theta\,dx,\ \theta-Mw=\varphi-1-2\cos(\theta_N^+-\varphi) \tag{2.11}$$

After eliminating w from the boundary condition on the interfacial boundary (2.1) and the integral (2.11), we obtain

$$(\theta+1-\varphi)\Delta-\cos\theta=-2\cos(\theta_N^+-\varphi),\ \Delta=\sin\varphi\cos\varphi_1/(\mu\sin\varphi_1) \tag{2.12}$$

The root $\theta=\theta_1$ of (2.12) determines the position of the characteristic NK and the solution in the domain MNK where the stresses and displacement velocity are constant (Fig.1). For an arbitrary point (x, y) of the domain GNK the solution is determined by the quantity θ, the root of the first equation of (2.9), that yields the position of the characteristic of the family (2.9) passing through the point (x, y).

The solution constructed holds in the case when the slope of the characteristic KN to the x axis is less than the slope of the characteristic GN and energy dissipation is positive at each point of the domain FNM.

We will investigate the change in θ in the domain FNM. We will examine the function $f(\theta)=(\theta+1-\varphi)\Delta-\cos\theta$. In the domain FNM $df(\theta)/d\theta=\Delta+\sin\theta>0$, since $\theta\in[\varphi, 2\varphi]$, in the domain FNG, and henceforth θ cannot exceed π and be less than 0 (for $\theta=0$ and $\theta=\pi$ the characteristic NK becomes parallel to the x axis and a slip zone is formed on the interfacial boundary /3, 4/).

We will evaluate the function $f(\theta)$ at the points θ_N^- and θ_1 and examine the case when $f(\theta_N^-)>f(\theta_1)$, i.e., when the following inequality is satisfied

$$2\cos(\theta_N^+-\varphi)(\Delta+\cos\varphi)+\Delta\,(1+\pi+\varphi-\theta_N^+)-\cos\theta_N^+>0 \tag{2.13}$$

Then since $df(\theta)/d\theta>0$, $\theta_N^->\theta_1$, i.e., the quantity θ decreases during motion along the characteristic (2.11) from the point G to the point K.

The characteristics (2.9) are inclined at an angle ψ to the x axis and

$$\operatorname{tg}\psi=-\sin\theta/(M+\cos\theta) \tag{2.14}$$

Differentiating relationship (2.14) with respect to the direction given by the vector $s=(\cos\varphi, \sin\varphi)$, we obtain

$$\frac{\partial}{\partial s}\operatorname{tg}\psi=-(1+M\cos\theta)(M+\cos\theta)^{-2}\left(\cos\varphi\frac{\partial\theta}{\partial x}+\sin\varphi\frac{\partial\theta}{\partial y}\right) \tag{2.15}$$

It follows from relationship (2.15) that the slope of the characteristics of the family (2.9), starting from the point N, to the x axis decreases if

$$\cos\theta+\sin\varphi>0 \tag{2.16}$$

It has been shown that $0\leqslant\theta\leqslant 2\varphi$ when condition (2.13) is satisfied and in this case inequality (2.16) is satisfied.

Let us examine the constraints that are imposed by inequality (2.13) on the incident wave intensity W_0. We set $\theta_N^+=\pi+\varphi^*$ $(0\leqslant\varphi^*\leqslant\varphi)$. The quantity θ_N^+ satisfies the boundary condition on the interfacial boundary ahead of the US front NE that has the form /5/

$$2W_0=\sin\varphi\,(1+\varphi-\varphi^*)+\mu\operatorname{tg}\varphi_1\cos\varphi^* \tag{2.17}$$

We will obtain the criterion for secondary plastic flow zone formation behind the US front from relationships (2.13) and (2.17)

$$W_0>\sin\varphi\cos(\varphi-\varphi^*)(1+\cos\varphi\Delta^{-1}) \tag{2.18}$$

The conditions for the EPH material to pass into the plastic state and for a slip zone to form ahead of the US front NE on the interfacial boundary, respectively, have the form /5/

$$2\ |\ W_0\ | \geqslant \sin\varphi\ (1+\Delta^{-1}\cos\varphi),\ 2\ |\ W_0\ | \geqslant \sin\varphi\ (1+\varphi+\Delta^{-1}) \tag{2.19}$$

Comparing the first inequality in (2.19) and condition (2.18) we can conclude that if $\varphi-\varphi^* \geqslant \pi/3$, then the plastic deformation ahead of the US front NE (Fig.1) always involves the formation of the secondary plastic flow zone behind the US front. If a slip zone is formed ahead of the wave front NE then $\varphi^*=0$. If $\varphi=\varphi_0$ is a root of the equation $2\cos\varphi-1-\varphi+\Delta^{-1}\cos 2\varphi=0$ (where $\varphi_0<\pi/4$), it follows from a comparison of the second inequality (2.19) and the criterion (2.18) that for $\varphi>\varphi_0$ slip zone formation ahead of the US front always involves the formation of a secondary plastic flow zone behind the wave front NE.

We will consider the energy dissipation D in the domain FNM. The condition of positivity of D in the plastic domain has the following form for the case being considered /3/:

$$D=\frac{1}{2}\left(\sin\theta\frac{\partial w}{\partial x}+\cos\theta\frac{\partial w}{\partial y}\right)\geqslant 0 \tag{2.20}$$

The stresses and rate of displacement are constant in the domains GNF and KNM; these are non-dissipative zeros and $D=0$ therein. The integral (2.11) holds in the domain GNM, from which we obtain

$$\partial\theta/\partial x=M\partial w/\partial x,\ \partial\theta/\partial y=M\partial w/\partial y \tag{2.21}$$

A fan of characteristics (2.9) is developed in the domain GNK and if we differentiate their equations, we determine the derivatives $\partial\theta/\partial x$, $\partial\beta/\partial y$. Using the expressions for these derivatives, we obtain from (2.21) and (2.20) that the condition for the energy dissipation to be positive in the domain FNM will be satisfied if

$$(1+M\cos\theta)(y\sin\theta-(x-x_N)\cos\theta)\geqslant 0$$

We have $\theta\in[0,2\varphi]$ in the domain GNK; consequently, the inequality obtained is satisfied. Therefore, $D\geqslant 0$ in the whole domain FNM.

The domain FNM in which the constructed solution holds is bounded on the left by the characteristic FM of the family (2.11). We will determine the equation of the line GK, the curvilinear part of the characteristic FM (Fig.1) (FG and KM are line segments, since the quantity θ takes constant values in the domains GNF and KMN). To do this we introduce the variable $z=y/(x-x_N)$ and, taking into account that the rectilinear characteristics (2.9) intersect the line GK (whose equation is $z=-\sin\theta/(M+\cos\theta)$), we obtain an ordinary differential equation to determine x from (2.11). Integrating this equation and satisfying the initial condition at the point G, we obtain the equation of the line GK in parametric form

$$\begin{gathered}x=C_1(M+\cos\theta)(1-\cos\theta)^\alpha(1+\cos\theta)^\beta+x_N,\ \alpha=(\sin\varphi-1)/4\\ y=-C_1\sin\theta\,(1-\cos\theta)^\alpha(1+\cos\theta)^\beta,\ \beta=-(\sin\varphi+1)/4\\ C_1=-x_N(\sin\theta_N^- -2\sin\varphi\cos(\theta_N^- -\varphi))(1+\cos\theta_N^-)^\beta(1-\\ \cos\theta_N^-)^{-\alpha}(2M\cos\theta_N^-)^{-1}\end{gathered} \tag{2.22}$$

3. We will examine the energy dissipation along the line FE behind the US front NE. In this case the condition for D to be positive has the form

$$D=\frac{1}{2}\left(\frac{\partial w^-}{\partial x}\sin\theta^-+\frac{\partial w^-}{\partial y}\cos\theta^-\right)\geqslant 0 \tag{3.1}$$

The equation of the line NE in parametric form is

$$\begin{gathered}x=x_N(1-\sin\varphi\cos\theta^+)/\Omega(\theta^+),\ y=x_N\sin\varphi\sin\theta^+/\Omega(\theta^+),\\ \Omega(\theta^+)=1+\sin(\theta^+-\varphi)\end{gathered} \tag{3.2}$$

Differentiating the function $w^-=w^-(x(\theta^+),y(\theta^+)),\ \theta^-=\theta^-(x(\theta^+),y(\theta^+))$ with respect to θ^+ and writing the equation obtained in conjunction with the system of equations of motion, we will have

$$\begin{gathered}\frac{\partial w^-}{\partial x}\frac{dx}{d\theta^+}+\frac{\partial w^-}{\partial y}\frac{dy}{d\theta^+}=\frac{\partial w^-}{d\theta^+},\quad \frac{\partial\theta^-}{\partial x}\frac{dx}{d\theta^+}+\frac{\partial\theta^-}{\partial y}\frac{dy}{d\theta^+}=\frac{d\theta^-}{dy}\\ \cos\theta^-\frac{\partial\theta^-}{\partial y}-\sin\theta^-\frac{\partial\theta^-}{\partial y}+M^2\frac{\partial w^-}{\partial x}=0,\quad \cos\theta^-\frac{\partial w^-}{\partial x}-\sin\theta^-\frac{\partial w^-}{\partial y}+\frac{\partial\theta^-}{\partial x}=0\end{gathered} \tag{3.3}$$

The derivatives

$$\frac{dx}{d\theta^+} = \frac{x_N(\sin\varphi - \cos\theta^+)}{\Omega^2(\theta^+)}, \quad \frac{dy}{d\theta^+} = \frac{-x_N\sin\varphi(\sin\varphi - \cos\theta^+)}{\Omega^2(\theta^+)}$$

$$\frac{dw^-}{d\theta^+} = -M(1 + 2\sin(\theta^+ - \varphi)), \quad \frac{d\theta^-}{d\theta^+} = -1$$

can be determined by using relationships (3.2), (2.4) and (2.7).

Consequently, system (3.3) should be considered as a system of linear equations in $\partial w^-/\partial x$, $\partial w^-/\partial y$, $\partial\theta^-/\partial x$, $\partial\theta^-/\partial y$, solving which and substituting the quantities $\partial w^-/\partial x$, $\partial w^-/\partial y$ into the inequality (3.1), we obtain after reduction that the condition for the energy dissipation to be positive will be satisfied if

$$\cos(\theta^+ - \varphi)\sin\varphi - (1 + 3\sin(\theta^+ - \varphi))\cos\varphi \geqslant 0 \tag{3.4}$$

But $\theta^+ \in [\pi, \pi + \varphi]$ /5/ consequently inequality (3.4) is not satisfied. Therefore, the material is in the elastic state in a certain neighbourhood of the line FE (Fig.1) behind the US front NE.

Following /6/, we will determine the range of variation of the secondary plastic loading wave (SPLW) velocity behind whose front secondary plastic deformation of the material occurs. The equations of the dynamics of an ideal elastic-plastic medium for antiplane deformation are presented in /3/, from which, by taking into account that the SPLW is a wave of weak discontinuity, we obtain

$$\left[\frac{\partial\tau_1}{\partial x_1}\right] + \left[\frac{\partial\tau_2}{\partial x_2}\right] - \rho_2\left[\frac{\partial w}{\partial t}\right] = 0, \quad \left[\frac{\partial\tau_i}{\partial t}\right] = \mu_2\left[\frac{\partial w}{\partial x_i}\right] - 2\mu_2\tau_2[\Lambda] \quad (i = 1, 2) \tag{3.5}$$

Here Λ is an undetermined positive multiplier in the associated flow law.

Using the geometric and kinematic compatibility conditions (3.5) can be written in the form

$$\left[\frac{\partial\tau_1}{\partial n}\right]\nu_1 + \left[\frac{\partial\tau_2}{\partial n}\right]\nu_2 + \rho_2 c_1\left[\frac{\partial w}{\partial n}\right] = 0, \quad c_1\left[\frac{\partial\tau_i}{\partial n}\right] = \mu_2\left(2[\Lambda]\tau_i - \nu_i\left[\frac{\partial w}{\partial n}\right]\right) \quad (i = 1, 2) \tag{3.6}$$

Here ν_i are projections of the vector normal to the SPLW on the coordinate axes, and c_1 is the SPLW velocity.

Eliminating the quantities $[\partial\tau_1/\partial n]$, $[\partial\tau_2/\partial n]$, in system (3.6) we obtain

$$\left[\frac{\partial w}{\partial n}\right](\rho_2 c_1^2 - \mu_2) + 2\mu_2[\Lambda](\tau_1\nu_1 + \tau_2\nu_2) = 0 \tag{3.7}$$

The flow condition is satisfied beyond the SPLW front, and differentiating it we obtain

$$\tau_1\frac{\partial\tau_1}{\partial n} + \tau_2\frac{\partial\tau_2}{\partial n} = 0 \tag{3.8}$$

Ahead of the SPLW front the material is in the elastic state; consequently

$$\tau_1\frac{\partial\tau_1}{\partial n} + \tau_2\frac{\partial\tau_2}{\partial n} \leqslant 0, \quad [\Lambda] = \Lambda^+ - \Lambda^- = -\Lambda^- < 0, \quad \Lambda^+ = 0 \tag{3.9}$$

Because the stresses are continuous in the SPLW, it follows from (3.8) and (3.9) that

$$\tau_1\left[\frac{\partial\tau_1}{\partial n}\right] + \tau_2\left[\frac{\partial\tau_2}{\partial n}\right] = \frac{[\Lambda]\Omega_1}{|c_1|}$$

Here Ω_1 is a certain non-negative quantity.

Using the last two equations of (3.6) to eliminate the quantities $[\partial\tau_1/\partial n]$, $[\partial\tau_2/\partial n]$, in the equation obtained, we obtain

$$[\Lambda](2\mu_2 k^2 - \Omega_1 c_1) - \mu_2[\partial w/\partial n](\nu_1\tau_1 + \nu_2\tau_2) = 0 \tag{3.10}$$

The system of two linear homogeneous Eqs.(3.7) and (3.10) in $[\Lambda]$, $[\partial w/\partial n]$ has a solution in the case when its determinant is zero, from which it follows that

$$\rho_2 c_1^2 = \mu_2 - 2\mu_2(\tau_1\nu_1 + \tau_2\nu_2)^2/(2\mu_2 k^2 - \Omega_1)$$

Since $\Omega_1 > 0$, we have $\rho_2 c_1^2 > \mu_2$ for $2\mu_2 k^2 < \Omega_1 < \infty$ and $0 < \rho_2 c_1^2 < \rho_2(c_p)^2 = \mu_2(1 - (\tau_1\nu_1 +$

$\tau_2\nu_2))^2/k^2$ for $0 < \Omega_1 < 2\mu_2 (1-(\tau_1\nu_1+\tau_2\nu_2)/k)^2$ (c_p is the plastic wave velocity). Therefore, the SPLW can propagate either more slowly than the platic waves or more rapidly than the elastic waves. Note that the SPLW can propagate only from the point F (Fig.1).

If the opposite is assumed, i.e., that the SPLW propagates from an arbitrary point of the segment GF, for instance from the point G, then we have a Cauchy problem for the elasticity theory equations on the segment GF whose solution will be constant (since the stresses and displacement velocity on GF take a constant value), which contradicts the condition on the shock NE and the solution in the plastic loading domain ahead of the wave front NE /5/.

4. We will consider an algorithm for constructing a SPLW which propagates in the case under consideration at a velocity that does not exceed the velocity of plastic wave propagation. Suppose we have a point L on the SPLW (Fig.1), i.e., we set $y_L = \delta$. We draw the characteristics LP and LR of the families (1.2) and (1.3) through the point L. Then $y_R = y_L = \delta$, $x_R = x_N - \varkappa\delta$. The solution is known at an arbitrary point (x, y) ahead of the US front NE /5/

$$\tau_1^+ = \sin\theta^+,\ \tau_2^+ = \cos\theta^+,\ w^+ = M^{-1}(1+\pi+\varphi-\theta^+),\quad y(M-\cos\theta^+) = x\sin\theta^+ \tag{4.1}$$

Relationship (1.3) holds along the line LR and using (4.1) and (2.2) we write it in the form

$$\tau_{1L} + w_L = \tau_{1R}^- + w_R^- = \tau_{1R}^+ + w_R^+ = \sin\theta_R^+ + \sin\varphi\,(1+\pi+\varphi-\theta_R^+) = f_1(\theta_R^+) \tag{4.2}$$

The characteristic LS of the family (1.4) that intersects the domain GNF (Fig.1), where the stresses and rate of displacement are constant, and determined from (2.10), passes through the point L. Consequently, the relationship along the characteristic LS takes the form

$$\theta_L + Mw_L = 1 + 3\varphi + (2(\pi - \theta_N^+ + \cos(\theta_N^+ - \varphi)) = C_1 \tag{4.3}$$

The stresses on the SPLW are continuous; consequently we set $\tau_1 = \sin\theta_L$, $\tau_2 = \cos\theta_L$. Then (4.2) and (4.3) are a system of equations to determine w_L and θ_L, which when solved yield

$$\theta_L = \theta^*,\ w_L = \sin\varphi\,(C_1 - \theta^*)$$

Here θ^* is the root of the equation $\theta_L - M\sin\theta_L + C_1 - Mf_1(\theta_R^+) = 0$.

Relationship (1.2) holds along the characteristic LP and by using (4.1) and (2.2), we obtain from it an equation for determining θ_p^+

$$\cos\theta_p^+ + \cos\varphi\,(1+\pi+\varphi-\theta_p^+) = \cos\theta_L^* + \cos\varphi\,(C_1 - \theta_L^*) \tag{4.4}$$

Let $\theta_p^+ = \theta_p^*$ be a root of (4.4); then the coordinates of the point P

$$x_p = x_N(1 - \sin\varphi\sin\theta_p^*)/\Omega(\theta_p^+),\ y_p = x_N\sin\varphi\sin\theta_p^*/\Omega(\theta_p^*)$$

are determined from the condition for the lines NE and PO to intersect.

The abscissa of the point L is determined from the equation of the characteristic LP

$$x_L = \varkappa(\delta - y_p) + x_p$$

The SPLW is constructed using the algorithm presented until the SPLW intersects the characteristic TG of the family (1.4). We note that because the solution is constant on the segment FG, the constant in the relationship along the characteristic (1.4) will be identical for all characteristics of this family that intersect the segment FG and, consequently, relationship (4.3) should be considered as an integral of the equations of motion in the domain TGF. The characteristic of the family (1.3) in the domain TGF are rectilinear, and the stresses and rate of displacement along them will remain constant.

Therefore, after having constructed the solution in the domain TGF, θ and w will be known on the characteristic TG. The initial conditions on the characteristic GM are given by (2.22). Therefore, we obtain a Goursat problem for the plasticity theory Eqs.(1.3) and (1.4) in the domain $TGMU$, which when solved will yield θ and w on the characteristic UM. The boundary condition (2.1) holds on the line MI and we obtain a mixed problem for Eqs.(1.3) and (1.4) in the domain UMI.

Further construction of the solution is carried out as follows. We specify the point J sufficiently close to the point T on the characteristic TI and draw a characteristic of the family (1.4) through it until it intersects the SPLW at the point Z (Fig.1). Because of the nearness of the points T, J, Z we consider the segment ZJ to be rectilinear. Furthermore, characteristics ZH and ZQ of the families (1.2) and (1.3) are drawn through the point Z. In the same way as above, by writing the equations of the characteristics jointly with relationships along these characteristics and equations governing the location of the point H and Q as points of intersection of the US NE and the appropriate characteristics of the family (1.5) issuing from the point O, we obtain

$$y_H\varkappa + x_H = x_N,\ y_z\varkappa + x_Q = x_N,\ (y_z - y_J)(M+\cos\theta_J) + (x_z - x_J)\sin\theta_J = 0 \tag{4.5}$$

$$y_z(M-\cos\theta_Q^+)-x_Q\sin\theta_Q^+=0,\ \varkappa(y_z-y_H)-x_z+x_H=0,$$
$$y_H(M-\cos\theta_H^+)-x_H\sin\theta_H^+=0$$
$$\varkappa w_z+\cos\theta_z=\cos\varphi(1+\pi+\varphi-\theta_H^+)+\cos\theta_H^+,\ Mw_z+\theta_z= Mw_J+\theta_J$$
$$w_z+\sin\theta_z=\sin\varphi(1+\pi+\varphi-\theta_Q^+)+\sin\theta_Q^+$$

The nine equations in (4.5) form a closed system for determining the nine unknowns y_H, x_H, y_z, x_z, x_Q, w_z, θ_H^+, θ_Q^+, θ_z, and finding its solution reduces to solving three equations in θ_H^+, θ_Q^+, θ_z.

Having determined the coordinates and the solution at the point Z, further construction of the solution is performed in the same way, i.e., a Goursat problem is solved in the domain $ZJIV$, etc.

We introduce the new variables

$$\xi=(\theta-Mw)/2,\quad \eta=(\theta+Mw)/2 \tag{4.6}$$

Assuming the Jacobian of the transformation Δ_1 to be non-zero, the solution of system (1.3) and (1.4) can be sought in implicit form $x=x(\xi,\eta)$, $y=y(\xi,\eta)$. Then we have from relations (4.6), (1.3) and (1.4)

$$\frac{\partial y}{\partial\eta}(M-\cos(\xi+\eta))=\sin(\xi+\eta)\frac{\partial x}{\partial\eta},\quad \frac{\partial y}{\partial\xi}(M+\cos(\xi+\eta))= -\sin(\xi+\eta)\frac{\partial x}{\partial\xi} \tag{4.7}$$

System (4.7) is linear, and boundary-value problems in the phase plane (ξ,η) are formulated on segments of the lines $\xi=\text{const}$, $\eta=\text{const}$. This explains the great attraction of system (4.7) for numerical integration as compared with system (1.3) and (1.4).

We will examine the condition for the energy dissipation to be positive in the phase plane where $x=x(\xi,\eta)$, $y=y(\xi,\eta)$. Differentiating these expressions with respect to x and y, we obtain a system of four equations which yields

$$\frac{\partial\xi}{\partial x}=\frac{1}{\Delta_1}\frac{\partial y}{\partial\eta},\quad \frac{\partial\xi}{\partial y}=-\frac{1}{\Delta_1}\frac{\partial x}{\partial\eta},\quad \frac{\partial\eta}{\partial x}=-\frac{1}{\Delta_1}\frac{\partial y}{\partial\xi},\quad \frac{\partial\eta}{\partial y}=\frac{1}{\Delta_1}\frac{\partial x}{\partial\xi} \tag{4.8}$$

when solved with respect to $\partial\xi/\partial x$, $\partial\xi/\partial y$, $\partial\eta/\partial x$, $\partial\eta/\partial y$.

It follows from (4.8), (4.7) and (2.20) that the condition for the energy dissipation to be positive in the phase plane (ξ,η) takes the form

$$\frac{1}{\Delta_1}\left[\cos(\xi+\eta)\left(\frac{\partial x}{\partial\xi}+\frac{\partial x}{\partial\eta}\right)-\sin(\xi+\eta)\left(\frac{\partial y}{\partial\xi}+\frac{\partial y}{\partial\eta}\right)\right]\geqslant 0 \tag{4.9}$$

The initial point of the unloading wave is determined by the violation of condition (4.9) on the interfacial boundary. Applying the algorithm elucidated in /3/ to determine the initial unloading wave velocity c_0, we obtain that the quantity c_0 is a root of the equation

$$\left[\frac{\partial(\eta-\xi)}{\partial x}+c_0\frac{\partial(\eta-\xi)}{\partial y}\right](1-c_0\varkappa^2\mu\,\mathrm{tg}\,\varphi_1)+ \left[\frac{\partial(\eta+\xi)}{\partial x}+c_0\frac{\partial(\xi+\eta)}{\partial y}\right]M(1-\mu\,\mathrm{tg}\,\varphi_1)\sin(\xi+\eta)=0$$

Here $\partial x/\partial\xi$, $\partial x/\partial\eta$ are approximated by finite differences, while the derivatives $\partial y/\partial\xi$ and $\partial y/\partial\eta$ are determined from (4.7). It is convenient to use the method of characteristics /7/ for a further construction of the unloading wave in the case under consideration.

5. The case when a slip zone is formed ahead of the US front NE should be examined separately, i.e., the characterstic OF becomes parallel to the x axis /5/. We introduce the system of polar coordinates

$$x=x_N+r\cos\psi,\ y=r\sin\psi \tag{5.1}$$

The system of equations describing the motion of the medium in the plastic domain takes the following form in the system of polar coordinates (5.1)

$$\cos\theta R(\theta)-\sin\theta R_1(\theta)+M^2R(w)=0,\ \cos\theta R(w)-\sin\theta R_1(w)+ R(\theta)=0 \tag{5.2}$$

$$B(f) = r\cos\psi\frac{\partial f}{\partial r} - \sin\psi\frac{\partial f}{\partial\psi}, \quad R_1(f) = r\sin\psi\frac{\partial f}{\partial r} + \cos\psi\frac{\partial f}{\partial\psi}$$

We will seek the solution of system (5.2) in the form of power series

$$\theta = \sum_{k=0}^{\infty}\theta_k(\psi)r^k, \quad w = \sum_{k=0}^{\infty}w_k(\psi)r^k \tag{5.3}$$

Substituting the expansion (5.3) into (5.2) and equating terms in r^k to zero, we obtain a system of two ordinary differential equations to determine the appropriate coefficients θ_k and w_k. For $k=0$ we obtain

$$\sin(\psi+\theta_0)\frac{d\theta_0}{d\psi} + M^2\sin\psi\frac{dw_0}{d\psi} = 0, \quad \sin\psi\frac{d\theta_0}{d\psi} + \sin(\psi+\theta_0)\frac{dw_0}{d\psi} = 0 \tag{5.4}$$

Firstly, the trivial solution

$$\theta_0 = A_0 = \text{const}, \quad w_0 = A_1 = \text{const} \tag{5.5}$$

satisfies system (5.4).

Secondly, when the determinant of system (5.4) equals zero, we have

$$\theta_0 = \arcsin(\pm M\sin\psi) - \psi, \; w_0 = \mp(\theta_0 - A_2)/M, \; A_2 = \text{const} \tag{5.6}$$

For $k=1$ we obtain

$$\cos(\theta_0+\psi)\theta_1\left(1-\frac{d\theta_0}{d\psi}\right) + M^2\left(w_1\sin\psi - \sin\psi\frac{dw_1}{d\psi}\right) - \sin(\psi+\theta_0)\frac{d\theta_1}{d\psi} = 0 \tag{5.7}$$

$$\cos(\theta_0+\psi)\left(w_1 - \theta_1\frac{dw_0}{d\psi}\right) + \theta_1\cos\psi - \sin(\psi+\theta_0)\frac{dw_1}{d\psi} - \sin\psi\frac{d\theta_1}{d\psi} = 0$$

to determine the coefficients θ_1, w_1.

In the case when $\theta_0 = \text{const}$, $w_0 = \text{const}$, the general solution of system (5.7) has the form

$$w_1 = A_3f^+ + A_4f^-, \quad \theta_1 = M[A_3f^+ - A_4f^+], \quad A_3 = \text{const} \tag{5.8}$$
$$A_4 = \text{const}, \quad f^\pm = M\sin\psi \pm \sin(\psi+\theta_0)$$

When relationships (5.6) are satisfied, the integrals of system (5.7) are

$$Mw_1 \mp 3\theta_1 = 0, \; \theta_1 \pm Mw_1 = A_i\sqrt{\sin\psi}\exp(\pm F(\psi)/M), \quad A_i = \text{const} \tag{5.9}$$
$$(i = 5, 6)$$
$$\left(F(\psi) = \int_0^{\psi}\frac{\cos(\theta_0+\psi)}{\sin\psi}d\psi\right)$$

We will confine ourselves to two terms of the series in the expansions (5.3).

The motion of the medium in the elastic domain and in the unloading zone is described by a system of equations /3/, that takes the following form in the system of polar coordinates:

$$R(\tau_2) + R_1(w) = 0, \; R_1(\tau_2) + \varkappa R(w) = 0 \tag{5.10}$$

We will seek the solution of system (5.10) in the form of power series

$$\tau_2 = \sum_{k=0}^{\infty}\tau_2^{(k)}(\psi)r^k, \quad w = \sum_{k=0}^{\infty}w^{(k)}(\psi)r^k \tag{5.11}$$

Substituting the expansions (5.11) into (5.10) and equating coefficients of r^k to zero, we obtain a system of ordinary differential equations to determine the coefficients $\tau_2^{(k)}$, $w^{(k)}$ and we will write its general solution as follows:

$$w^{(k)} = \varkappa^{-1}[T_kf_k^- + B_kf_k^+], \quad \tau_2^{(k)} = T_kf_k^- - B_kf_k^+ \tag{5.12}$$
$$f_k^\pm = (\cos\psi + \varkappa\sin\psi)^k, \quad T_k = \text{const}, \quad B_k = \text{const}$$

Using expansion (5.11) we have from relationship (1.3)

$$\tau_1 = \sum_{k=0}^{\infty} r^k (L_k(\psi) - w^{(k)}(\psi)) \tag{5.13}$$

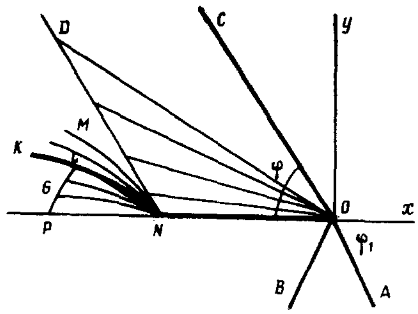

Fig.2

6. The constants in relationships (5.5), (5.6), (5.9), (5.12) and (5.13) are determined from the conditions connecting the integrals obtained in the neighbourhood of the point N (Fig.2). Expanding the solution ahead of the US front and the jump in the velocity w in a power series in r in the neighbourhood of the point N (taking two terms in the expansions into account here), we will have

$$\begin{gathered} \theta^+ = \theta_0^+ + \theta_1^+ r = \pi - r\Delta_3, \quad \tau_1^+ = \sin\theta_0^+ + \cos\theta_0^+\theta_1^+ r = r\Delta_3 \\ w^+ = M^{-1}(1 + \pi + \varphi + \theta^+) = (1+\varphi)/M + r\Delta_3/M, \ [w] = E_0 + E_1 r \\ \tau_2^+ = \cos\theta_0^+ - \sin\theta_0^+\theta_1^+ r = -1, \quad \Delta_3 = (1 + \sin\varphi)/x_N \end{gathered} \tag{6.1}$$

We have $\psi = \pi - \varphi$ on the line ND. Connecting the elastic solution (5.11)-(5.13) on the line ND in the neighbourhood of the point N (taking two terms in the expansion into account) and the solution in the plastic domain ahead of the US front (6.1) taking relationships on the line of strong discontinuity (2.2) into account, we obtain

$$\begin{gathered} L_0 + L_1 \sin\psi r - \varkappa^{-1}(T_0 + B_0) + 2\varkappa^{-1}\cos\varphi T_1 r = E_0 + E_1 r + \Delta_3 r \\ \varkappa^{-1}(T_0 + B_0) + \varkappa^{-1} 2\cos\varphi T_1 r + (1+\varphi)/M + \Delta_3 r/M = E_0 + E_1 r \\ T_0 - B_0 - 2\cos\varphi T_1 r = \varkappa(E_0 + E_1 r) - 1 \end{gathered} \tag{6.2}$$

For $r = 0$ the flow condition (2.3) should be satisfied at the point N. Taking this into account and equating the coefficients of identical powers of r to zero in relatinships (6.2), we obtain a system of linear equations to determine $L_0, L_1, T_0, T_1, B_0, E_0, E_1$, from which

$$\begin{gathered} L_0 = (1+\varphi)/M, \ L_1 = (1 + \sin\varphi)\Delta_3/\sin\varphi, \ T_0 = ((1+\varphi)\varkappa M^{-1} - 1)/2 \\ B_0 = -2\varkappa^2/M^2 + {}^1/_2 + \varkappa(1+\varphi)/(2M), \ E_0 = 2\varkappa/M^2, \ E_1 = \sin\varphi\Delta_3/2, \\ T_1 = -\Delta_3/4 \end{gathered} \tag{6.3}$$

A fan of curvilinear characteristics (1.4) whose inclination to the x axis should be negative is developed in the neighbourhood of the point N. We obtain from relatinships (1.4), (6.3) and (5.6) (with the upper sign)

$$\operatorname{tg}\psi^* = \sin\theta_0(M - \cos\theta_0) = 2\varkappa(M^3 - M^2 + 2) > 0$$

Therefore, the integral (5.6) with the lower sign holds in the case under consideration: using the latter and relationships (1.4) and (6.3) we obtain

$$\operatorname{tg}\psi^* = -\sin\theta_0/(M + \cos\theta_0) = -2\varkappa(M^3 + M^2 - 2) < 0 \tag{6.4}$$

The quantity ψ^* given by (6.4) governs the position of the first characteristic of the fan NM (Fig.2). The constant in the second relationship in (5.6) (with the lower sign) is determined from the relationships (6.2), (2.4) and (2.7)

$$Mw_0 - \theta_0 = 1 + \varphi - 2\cos\varphi \tag{6.5}$$

Using (6.5), the boundary condition on the interfacial boundary (2.1) can be represented in the form

$$1 - \varphi - 2\cos\varphi + \theta_0 = M\mu \operatorname{tg}\varphi_1 \cos\theta_0 \tag{6.6}$$

The root $\theta_0 = \theta_0^*$ of (6.6) determines the inclination to the x axis for the last characteristic of the fan NG (Fig.2)

$$\operatorname{tg}\psi_0^* = -\sin\theta_0^*/(M - \cos\theta_0^*) \tag{6.7}$$

The fan of curvilinear characteristics issuing from the point N will be cut off by the SPLW NK (Fig.2). Let the equation of the SPLW have the form $\psi = \psi(r)$. Expanding the function $\psi(r)$ in a power series in r and confining ourselves to two terms of the series, we have

$$\psi = \psi_0 + \psi_1 r \tag{6.8}$$

Connecting the elastic solution (5.11)-(5.13) on the SPLW front (6.8) in the neighbourhood of the point N (we limit ourselves to two terms in the expansions) and the plastic solution (5.6) and (5.9) (with lower signs), and taking into account here that the stresses and rate of displacement are continuous on the SPLW front, we obtain a system of three equations. Hence, equating coefficients of identical powers of r to zero, we obtain a system of linear equations to determine B_1, A_2, A_5, ψ_0, ψ_1, which we solve to find

$$B_1 = \frac{\sin\varphi\cos\varphi}{\sin(\varphi+\psi_0)}\left(F_1 - \frac{8\sin\varphi\,(F_2 - 2F_1)}{\Delta_2}\right], \quad F_1 = \frac{T_1\sin(\varphi-\psi_0)}{\sin\varphi\cos\varphi} \tag{6.9}$$

$$A_5 = \frac{4}{\Delta_2\sqrt{\sin\psi_0}}\,[2F_2(1+\cos 2\varphi) + F_1(2\sin\varphi - 1)], \quad F_2 = L_1\sin\psi_0$$

$$\psi_1 = F_3(\psi_0)\left[\frac{F_2(3+2\sin\varphi) + 2(\cos 2\varphi - 3)F_1}{\Delta_2}\right],$$

$$F_3(\psi) = \left(1 + \frac{M\cos\psi}{\sqrt{1-M^2\sin\psi}}\right)^{-1}$$

$$\psi_0 = \operatorname{arctg}(-\sin 2\varphi/(M+\cos 2\varphi)), \; A_2 = \varphi - 1 + 2\cos\varphi - \psi_0,$$

$$\Delta_2 = -4\,(2\sin\varphi + \cos 2\varphi)$$

Connecting the solution (5.5), (5.8) to the solution (5.6), (5.9) in a similar way in the neighbourhood of the last characteristic of the fan NG (Fig.2), for which the expansion holds (where the quantity ψ_0^* is determined from the relationship (6.7)) $\psi^* = \psi_0^* + \psi_1 r$, we find the remaining constants

$$A_0 = \arcsin(-M\sin\psi_0^*) - \psi_0^*, \; \psi_1^* =$$

$$-A_5\sqrt{\sin\psi_0^*}\exp(-F(\psi_0^*)/M)F_3(\psi_0^*)/4$$

$$A_4 = -\frac{A_5\exp(-F(\psi_0^*)/M)}{(2M)^2\sqrt{\sin\psi_0^*}}, \quad A_3 = \frac{A_4(1+\mu\operatorname{tg}\varphi_1\sin A_0)}{1-\mu\operatorname{tg}\varphi_1\sin A_0},$$

$$A = (A_0 - A_2)/M$$

Therefore, the solution is completely defined in the neighbourhood of the point N. Later, by specifying a point L lying sufficiently close to the point N (Fig.2) on the SPLW and drawing a characteristic LP of the family (2.8) through it, we can construct the solution according to the algorithm elucidated earlier for the case when there is no slip zone ahead of the US front.

The process of plastic deformation behind a reflected US front was investigated in /8-10/.

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MOTIONS DOUBLY ASYMPTOTIC TO INVARIANT TORI IN THE THEORY OF PERTURBED HAMILTONIAN SYSTEMS*

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Poincaré's theory /1/ of the formation of isolated periodic motions during the perturbation of resonant invariant tori of integrable Hamiltonian systems was generalized in /2/ by the methods of KAM-theory to the case of conditionally-periodic motions. In this paper variational methods are used to prove the existence of motions doubly-asymptotic to the nascent invariant tori. The existence of such trajectories is important in the qualitative investigation of a perturbed system. For example, if the doubly-asymptotic trajectory is isolated, then the perturbed system is non-integrable /3/ and possesses stochastic behaviour. Arnol'd's /4/ diffusion for Hamiltonian systems with many degrees of freedom is based on the existence of motions doubly-asymptotic to invariant tori.

Let the Hamiltonian function $H = H_0 + \varepsilon H_1 + O(\varepsilon^2)$ of an autonomous Hamiltonian system with m degrees of freedom depends smoothly on the parameter ε. We assume that the unperturbed system with Hamilton function H_0 has a smooth compact invariant m-dimensional Lagrangian manifold M (a manifold M is Lagrangian if the restriction to M of the phase space's canonical 2-form is zero), entirely filled with n-dimensional invariant tori carrying conditionally-periodic motions with identical vector frequencies $\omega \in \mathbf{R}^n$. This means that a free action of the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/2\pi Z^n$ is specified on M: $\varphi \in \mathbf{T}^n$, $x \in M \to f(\varphi, x) \in M$, and for any $x \in M$ the curve $t \to f(\omega t, x)$ is a trajectory of the unperturbed Hamiltonian system. A principal example is the case /2/ when the unperturbed system is fully integrable, and M is its m-dimensional resonant torus, such that the corresponding frequency vector $\Omega \in \mathbf{R}^m$ satisfies $m-n$ resonance relations of the form $\langle k, \Omega\rangle = 0$, $k \in \mathbf{Z}^m$.

A neighbourhood of the Lagrangian manifold M in the phase space can be identified with a neighbourhood of the set $\{y = 0\}$ in the cotangent bundle $T^*M = \{(x, y): x \in M, y \in T_x^*M\}$ with canonical 2-form $dx \wedge dy$. We extend the action of the torus T^n on M to the Hamiltonian action of $\mathbf{T}^n$ on T^*M:

$$\varphi \in \mathbf{T}^n,\ (x, y) \in T^*M \to (f(\varphi, x), f_x^{*-1}y) \tag{1}$$

Let $\bar{H}_0$ and $\bar{H}_1$ be the results of averaging the functions H_0 and H_1 with respect to the action (1), for example

$$\bar{H}_0(x, y) = \frac{1}{(2\pi)^n}\int_{\mathbf{T}^n} H_0(f(\varphi, x), f_x^{*-1}y)\, d\varphi \tag{2}$$

We make the following assumptions:

1) the frequency vector ω is non-resonant in the sense of KAM-theory: there exist $C > 0$ and $N > n-1$ such that

$$|\langle\omega, k\rangle| \geqslant C\|k\|^{-N} \tag{3}$$

for all non-zero $k \in \mathbf{Z}^n$.

2) the following convexity constraint is satisfied: the Hessian $A(x) = \bar{H}_{0yy}(x, 0)$ is positive-definite for all $x \in M$. This condition can be weakened, for example, by changing it to the

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